Exercise sheet

Transversality and its applications. GQT minicourse

August 23, 2022

1 Weird functions and Sard

Exercise 1. Let $K \subset \mathbb{R}$ be a closed subset. Prove that there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ whose zero set is exactly K. Strategy:

- Observe that the distance function $g : \mathbb{R} \to \mathbb{R}$ to K is continuous.
- Consider the sets $K_n := \{x \in \mathbb{R} \mid g(x) \geq 1/n\}$, where $n \in \mathbb{Z}^+$. Construct a function f_n that is identically 1 on K_n but zero on the complement of K_{n+1}. **Hint:** Convolve the indicator function of K_n against a suitably chosen bump function.
- Find constants a_n so that $f := \sum a_n f_n$ is the desired function. **Note:** To prove that f is smooth, you have to choose the a_n carefully.

Deduce that there is a smooth function on $\mathbb{R}$ whose critical points are exactly K.

Exercise 2. Let $U \subset [0,1]$ be an open dense subset with zero measure complement K. Prove that there is a C^1 function $f : [0,1] \to [0,1]$ whose critical *values* are exactly K. **Hint**: Consider a homeomorphism $\chi : [0,1] \to [0,1]$ that is smooth, a diffeomorphism when restricted to $(0,1)$, and has all derivatives vanishing at the endpoints. In each component of U, use a suitable reparametrisation of χ.

Exercise 3. Let $K \subset \mathbb{R}$ be a subset such that $K + K$ contains a non-empty open subset of $\mathbb{R}$. Prove that there is no C^2-function $f : \mathbb{R} \to \mathbb{R}$ whose singular values contain K. **Hint:** Construct a suitable function $\mathbb{R}^2 \to \mathbb{R}$ and apply Sard to it.

Exercise 4. Let C be the Cantor set. Prove that $C + C = [0, 2]$.

Exercise 5. Find a C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ whose set of critical values has positive measure.

Exercise 6. Let M be a smooth manifold. Let $f : M \to \mathbb{R}$ be a C^1-function with locus of critical points $A \subset M$. Prove that f is constant along any piecewise C^1 curve $\gamma \subset A$.

Note: In "*A function not constant on a connected set of critical points*", Whitney constructs a C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ that is not constant over path-components of the critical set. The caveat is that the critical set A of f is a non-rectifiable curve (i.e. we cannot connect all points in A by piecewise C^1 paths).

Exercise 7. Let M and N be m and n dimensional, respectively, with $m < n$. Determine the range of $\delta \in (0, 1]$ such that the image of any $f : M \to N$ of Hölder regularity δ has measure zero.

2 Handle dimension and transversality

Exercise 8. A *triangulation* of a manifold N is a homeomorphism $f : K \to N$ from a simplicial complex. In Differential Topology, we furthermore assume that f is smooth when restricted to all simplices of K. You can prove that every smooth manifold N admits a triangulation. A possible strategy, due to Whitney, reads:

- Embed N into some euclidean space E.
- Produce a sufficiently fine triangulation T of E.
- In a tubular neighbourhood of N, make simplices transverse to N by applying transversality.
- Choose a nice subcomplex K of T, homotopy equivalent to N.
- The projection of K to N should be a homeomorphism.

You probably do not want to do this in detail. I recommend that you think about it, draw some pictures when N is a curve/surface embedded in $\mathbb{R}^3$, and move on.

Exercise 9. We say that N has *handle dimension* at most n if it deformation retracts to an n-dimensional CW-complex $K \subset N$, called the *skeleton*, whose simplices are smoothly embedded.

- Find an example in which the handle dimension is smaller than the dimension, and the skeleton (giving that handle dimension) cannot be chosen to be a manifold.
- Prove that, if N is closed, its handle dimension is equal to its dimension. **Hint:** Cohomology.
- The following is very difficult but, if you have plenty of time: prove that every open manifold M of dimension m has handle dimension at most $m-1$. **Idea:** Pick a triangulation and try to push M into a neighbourhood of the codimension-1 simplices.

Exercise 10. Fix an m-dimensional manifold M. Consider N and L, submanifolds of handle dimensions n and l, respectively. Prove that, generically, $N \cap L = \emptyset$ if $n + l < m$. **Hint**: The claim follows once the skeleta are made disjoint. To prove this latter fact, use induction on the handle dimension and apply transversality to the cells.

Exercise 11. Let M be a manifold of handle dimension m and dimension n. Prove that M immerses into $\mathbb{R}^{\max(n,2m)}$ and embeds into $\mathbb{R}^{\max(n,2m+1)}$.

3 Bordism

Let us recall the construction of (unoriented, singular) bordism. The idea is to build a homology theory whose building blocks are manifolds and not simplices. We fix an m-dimensional manifold M (the upcoming definitions work for topological spaces as well, but then we would not have transversality).

We define $Z_n = Z_n(M)$ to be the set whose elements are pairs $(N, f : N \to M)$, with N a closed n-dimensional manifold and f a smooth map. Disjoint union defines a notion of addition in Z_n. An element in Z_n is *nullbordant* if it is of the form $(\partial W, g|_W)$, where W is an $(n+1)$-manifold with boundary and $f : W \to M$ is a smooth map. This defines a equivalence relation in Z_n: two elements are *bordant* if their disjoint union is nullbordant. We denote the resulting set of equivalence classes by B_n; addition descends to this quotient.

Exercise 12. We now use transversality to define an intersection pairing. Given $[N, f] \in B_n$ and $[L, g] \in B_l$, we define their intersection to be an element $[N, f] \cap [L, g] \in B_{n+l-m}$:

- Given $(f, g) : N \times L \to M \times M$, use transversality to find $h : N \times L \to M \times M$ close to (f, g) and transverse to the diagonal Δ_M.
- Define $[N, f] \cap [L, g]$ to be $[I := h^{-1}(\Delta_M), h|_I]$. Prove that this is well-defined (i.e. it does not depend on choices and it only depends on (N, f) and (L, g) up to bordism).

- Let N be m-dimensional and L be a point. How does $[N, f] \cap [L, g]$ relate to the degree of f?

Exercise 13. We can define oriented singular bordism by considering now pairs (N, f) with N oriented. A nullbordant element $(\partial W, g|_W)$ is taken to have the boundary orientation given by the orientation of W. Prove that the intersection pairing can be defined in this oriented setting as well. How does this relate to the degree of a map?

4 Transversality in geometry

Exercise 14. A symplectic structure in a $2n$-dimensional manifold M is a closed, non-degenerate 2-form ω. Non-degeneracy means $\omega^n \neq 0$ (i.e. at each point, ω, seen as a matrix, has non-zero determinant). Prove that a generic closed 2-form is symplectic in the complement of a smooth hypersurface $H \subset M$. **Hint:** Perturb ω using exact 2-forms.

Exercise 15. A 1-form α in a $(2n+1)$-dimensional manifold M is said to be a contact structure if $\alpha \wedge d\alpha^n \neq 0$ (i.e. $d\alpha$ is non-degenerate when restricted to the hyperplane field $\ker(\alpha)$). Prove that a generic 1-form is contact in the complement of a smooth hypersurface $H \subset M$.

Exercise 16. A 1-form α in a $(2n+2)$-dimensional manifold M is said to be an even-contact structure if $\alpha \wedge d\alpha^n \neq 0$. Prove that a generic 1-form is even-contact in the complement of a discrete collection of points $A \subset M$. Deduce that, if M is open, it has an even-contact structure. **Hint:** Its handle dimension is at most $2n + 1$.

Exercise 17. Let M be a manifold. Prove that a generic symmetric 2-tensor g on M is non-degenerate in the complement of a stratified hypersurface $H \subset M$. What is a reasonable stratification for H (and what are the dimensions of the strata generically)? **Hint:** Think first about the linear case (i.e. stratify the space of symmetric bilinear maps on a vector space).

5 Jet spaces

Exercise 18. Prove that the space of immersions is open (as a subset of the space of all maps in the strong C^1-topology). Then, you may want to prove the same statement for embeddings (but it is much trickier!) Prove that the space of embeddings is not open in the C^0-topology.

Exercise 19. Let us introduce coordinates (x, y, z) on $J^r(\mathbb{R}^n, \mathbb{R}^m)$. Here $x \in \mathbb{R}^n$ are the domain variables and $y \in \mathbb{R}^m$ the target variables. Given a multi-index $I = (i_1, \cdots, i_n)$ of cardinality $|I| = i_1 + \cdots + i_n \leq r$, we write z_j^I for the variable representing the derivative $\partial_1^{i_1} \cdots \partial_n^{i_n}$ of y_j with respect to the x-variables.

Given diffeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^m \to \mathbb{R}^m$, we can produce a diffeomorphism of jet space $\Phi : J^r(\mathbb{R}^n, \mathbb{R}^m) \to J^r(\mathbb{R}^n, \mathbb{R}^m)$ by setting

$$\Phi(j^r_p f) := j^r_{\phi(p)}(\psi \circ f \circ \phi^{-1}).$$

We call this a *point symmetry.* When ψ is taken to be the identity, we call this a *domain symmetry.*

- Prove that point symmetries in $J^1(\mathbb{R}, \mathbb{R})$ are indeed diffeomorphisms.
- Prove that the only subsets $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R})$ invariant under point symmetries are $J^1(\mathbb{R}, \mathbb{R})$ itself and the empty set.
- Prove the analogous statements for $J^r(\mathbb{R}^n, \mathbb{R}^m)$.

Exercise 20. Suppose that $\mathcal{R} \subset J^r(\mathbb{R}^n, \mathbb{R}^m)$ is invariant under domain symmetries. Prove that $\pi : \mathcal{R} \to \mathbb{R}^n$, the projection to the x-variables, is a fibre bundle. Can you characterise all such $\mathcal{R}$ in the case $n, m, r = 1$?

Exercise 21. Find a subset $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R})$ such that:

- There is a section $F : \mathbb{R} \to \mathcal{R}$ (here the domain corresponds to the x-variable).
- There is no function $f : \mathbb{R} \to \mathbb{R}$ such that $j^1 f$ takes values in $\mathcal{R}$.

Hint: You can pick F first and then construct a suitable $\mathcal{R}$.

Can you find an example with $\mathcal{R}$ open and invariant under domain symmetries?

6 Random exercises

Exercise 22. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\partial^j f(0) = 0$ for $j < r$ and $\partial^r f(0) \neq 0$. Then, the Malgrange preparation theorem states that there is g non-vanishing such that $f(x) = g(x)x^r$ close to zero.

Deduce that there is a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$, locally defined close to the origin, such that $f \circ \psi = \pm x^r$. We say that, thanks to the coordinate change ψ, we have put f in *normal form* close to zero.

Prove that there is no normal form for functions flat at zero (i.e. smooth functions with $\partial^j f(0) = 0$ for all j). I.e. exhibit various flat functions that cannot be transformed into one another by a local change of coordinates.

The punchline here is that, locally, smooth functions resemble analytic functions when their Taylor polynomials are non-vanishing, but may be very wild otherwise.

Exercise 23. Let M be the vector space of m-by-n matrices, $m \leq n$. Write $M_k \subset M$ for the locus of matrices of rank exactly $k \leq m$. Recall that the general linear group GL_m acts on M by right-multiplication. Similarly, GL_n acts on M from the left.

- Prove that there is a transitive action of $\mathrm{GL}_m \times \mathrm{GL}_n$ on M_k.
- Compute the isotropy of this action (i.e. pick an element in M_k and determine the subgroup H fixing it).
- Deduce that M_k is smooth and diffeomorphic to the quotient $(\mathrm{GL}_m \times \mathrm{GL}_n)/H$.

Exercise 24. Let M be the space of m-by-n matrices; let $m \leq n$. Consider now $A_k \subset M$, the locus of matrices of rank *at most* $k \leq m$. Observe that this is an algebraic subvariety. Find examples of m, n and k such that A_k is not smooth.