TEACHING PORTFOLIO

Documentation supporting my application for the Basiskwalificatie Onderwijs (BKO)

Álvaro del Pino

Deparment of Mathematics
Utrecht University

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Personal essay

A look at my former and future teaching

My experience as a post-doc has been a experience of many "firsts". For the first time, I have tutored master students as they tipped their toes on the waters of research. I have shared my ideas with my first PhD students and, as I have done this, I have seen them mature into extremely competent mathematicians from whom I have also learnt a great deal. I have assumed, for the first time (or rather, co–assumed together with another lecturer), the task of giving a full course for master students, along with the rather daunting task of developing a set of notes for it.

As I have done these things, I have tried to look critically at my performance: Are my lectures structured correctly and do they complement the existing notes? Are the students receiving from me feedback that is useful and that they can use to improve? Is my interaction with them conductive to learning? Do my notes and exercises provide the correct steps to build up proficiency? Am I addressing the differences in background of my students suitably so that they all benefit from the course/project? In the case of my PhD students, am I providing adequate support for the next steps in their career, not only mathematically but professionally?

I started formulating these questions about my teaching while I was attending the "Teaching in Higher Education" (THE) course at Universiteit Utrecht. The weekly meetings helped me to structure these concerns better, and to bring to light issues I had not considered before. I believe this has allowed me to produce a strong portfolio that justifies my application for a Basiskwalificatie Onderwijs (BKO).

In this personal essay I will look at the teaching concerns I have just voiced, and at the issues I have observed when self-reflecting about them. My goal is to focus on three key aspects that I want to improve on as I progress as a docent.

Different students require different approaches

Most of the students I have interacted with at this stage have been part of the Geometry track. Despite of this, they form an extremely ample spectrum, having differing degrees of motivation, disparate strengths and weaknesses, and very different mathematical backgrounds, interests, and manners of thinking.

Early in my Symplectic Geometry lectures I noticed that many of the students had extremely strong backgrounds, and were able to easily follow the classes in their entirety (despite them being 3 hours long). They were also interested in the material and willing to step up when a more challenging exercise appeared. At the same time, I observed a couple of students slowly slipping, handing-in progressively weaker assignments as the course progressed and the gaps in their knowledge became greater.

These two groups require different approaches, different methods of teaching. Let me go over the latter group first.

Early in the course I decided that I would create an atmosphere of active learning during the class to help those struggling. Instead of doing all of the examples myself, I set-up in-class activities to force them to think about the material covered up to that point and how it may be applied. Implementing this idea has been a learning experience for me: I have observed that it works well in simple situations where the students are given plenty of time to think. For instance: coming up with examples of a new definition, or solving somewhat mechanical computations. That is, it is effective to activate their low level knowledge and to make them more familiar with the material. At the same time, I have had a couple of failures when the task required higher level thinking grounded on higher proficiency (that they did not have yet): most of the students were simply unable of making any progress. I would like to keep experimenting with this idea in future courses. Particularly, I would like to improve my ability to guide their thinking by posing effective questions.

Another important aspect to remember, in order to help those struggling, is that following a lecture in Mathematics can be, by itself, active learning: Understanding each of the steps involved in an argument requires that the listener takes an active role. As such, as a lecturer, I must try to make this task as easy as possible for the student. For instance, I should pace myself, giving the students time to process new information by having moments of silence when an idea is complete. Key ideas should jump at them, by being clearly marked as such in the blackboard. I believe having a helpful set of notes, that they can print and bring to the lecture, is a first step in this direction.

At the opposite end of the spectrum, one must keep the course interesting for the over-achieving students as well, by providing them with a challenge. A mistake I made (as pointed out previously) was to propose demanding in-class activities based on material we had just seen. This is very far away from how understanding in Mathematics works: proficiency requires time. In fact, at the end of the course, one of the students told me he felt more comfortable thinking at home, without the pressure. On the next iteration of the course, I would like to try an inverted approach: to have them work on something more elaborate as a preparation for the lecture and then spend time revisiting it during the class. This can be done with different scopes: either solving an exercise from one session to the next, or perhaps a small (optional?) project requiring several weeks to complete with a scheduled evaluation in the middle to provide them with feedback.

What is the purpose of an assignment?

One of the most frustrating aspects for the students of the Symplectic Geometry course were the assignments. Having a weekly deadline can feel like an insurmountable pressure, particularly when one struggles with one of the exercises. Even if we, as lecturers, are willing to be flexible with the hand-in dates, the students are often reticent to ask for this flexibility. Having to hand-in problems from one session to the next implies that there is no in-class time in which they may ask about the assignments. These are issues I have observed myself and that students have rightfully pointed out in their evaluations.

What is important here is to realise that the purpose of the assignments is not to grade the students. This is readily apparent from the fact that they amount for an extremely little portion of the final grade. Instead, they are meant to activate the learning process of the student, reinforcing and complementing what is seen in class. It may be true that having weekly hand-ins forces them to look at the material before they attend the next lecture, but that is precisely the issue: it is an imposition. It does not take into account the personal situation of each of them. It does not give them the freedom of choosing how or when they learn.

Next year, I would like to implement a block by block scheme. That is, the students will be told at the beginning of a thematic block what the exercises are that they must solve. As the lectures progress, I will point out how the material is relevant for each exercise, and I will encourage them to look at those activities that they may already solve with what we have seen. This will allow them to ask for clarification during the classes and to look at the material at their own pace. Additionally, having this extra time might allow for more involved activities as opposed to disconnected exercises. Being this the case, the hand-ins could make up a larger portion of the final grade, with room for bonus points based on extra performance.

Coaching: writing and presenting

In my role as an advisor/tutor, I often get carried away by the mathematical content. I am infinitely marveled by the discussion of ideas and clever arguments. However, a large part of what we do in our professional lives relies on soft skills. It is important to be able to convey all these beautiful ideas through the talks we give and the papers we write. And yet, I believe this is an aspect I have neglected when I coach others.

I have tutored master students during the so-called “Orientation in Mathematical Research” (OMR) course that they must take. The students must carry out a small project in which they get a small taste of how research works. Having done it twice, on the second iteration I tried to emphasise the research aspect: I proposed a topic that required a somewhat limited background, but that would allow them to explore new ideas. In that sense, it was a big success, because they were able to actually come up with new results that some of them would like to pursue further.

Despite of this, I was not satisfied with the document they produced and the talk they gave. Both of them seemed like an afterthought, completely secondary to the mathematical content. I had been unable to convey to them that these were also important facets that they had to worry about.

I believe there were two problems from my part. Firstly, I did not give them clear parameters at the beginning. I did not inform them about how their work would be graded. Since I excitedly jumped straight into the Mathematics, so did they. Secondly, upon seeing a preliminary version of their project, I did not provide surgical feedback: I gave them too many pointers and it was unclear to them what the most important aspects to work on were.

I hope to be part of the OMR course later this year as well. I will keep what has worked so far, by proposing a project that allows the students to be creative. However, I will provide a more structured environment for them to work. At the beginning, I would like to give them a rubric of how I will grade, stressing the importance of the writing and presenting. This rubric will also guide me when I provide feedback during the project: it will be easier to compare where they are currently, with where they should be when the project comes to an end.

Final remarks

Next year I will be teaching, for the very first time, a bachelor course as the sole lecturer. This is, partly, why I decided to focus on the first two aspects I discussed. Working with a bigger and more heterogeneous group of students will certainly present a new challenge for me, but I think I am asking myself the right questions to approach it. What is certain is that I will be going into the experience with plenty of enthusiasm.

Course redesign

A critical look at the Mastermath Symplectic Geometry

Symplectic Geometry is one of the courses offered by Mastermath, the national programme for master studies in Mathematics. The course aims to introduce the students to the basic theory of *symplectic structures*. These are one of the most studied structures in Geometry, having their origin in Physics and, more concretely, in the Hamiltonian formalism of Mechanics.

The course is taught in 16 weekly sessions, each of which consists of 3 hours of lectures. We expect the students, on average, to spend 10 hours at home every week reviewing the material and working on the exercise sheets we provide.

Before the academic year 2018/19, the course was taught repeatedly by Federica Pasquotto and Fabian Ziltener. I replaced Federica as lecturer in Spring 2019, and I will be teaching the course for a second time on Spring 2020.

It is possible to divide the course in 5 thematic blocks that depend on one another in a somewhat linear fashion:

1. Symplectic Linear Algebra.
2. Symplectic structures and their automorphisms.
3. Submanifolds and normal forms.
4. Hamiltonian Lie group actions and reduction.
5. Contact Geometry.

Fabian and I decided that I would teach blocks (1), (3), and (5) and he would teach the other two. This actually divides the material in two halves of approximately the same length (8 sessions each).

2.1 Original goal of the redesign (3rd February 2019)

2.1.1 What do I want to change?

This course is taught every two years but this is my first time lecturing it. As such, my aim is twofold:

- I want to reshape certain parts to cover topics more topological in nature.
- I want to restructure the way in which it is taught. In particular, I want make more emphasis on in-class activities.

In particular, I want to add more depth to the existing set of notes. These were written by Professors Pasquotto and Ziltener for the previous iteration of the course.

2.1.2 Output

My intention is to rewrite the half of the **course notes** corresponding to the classes I will teach. I have already identified several points where I want to introduce changes (either because I would approach things differently or because I think there are important topics that are missing and I would like to address).

Apart from the notes themselves, I will develop an **accompanying teaching script** for some of the sessions.

Further, I want to create a more **appealing Course Guide**, in which learning objectives and prior requirements are clear.

2.1.3 Remarks

It is worth noting that math classes are usually given on a blackboard, instead of using slides. This is suitable because most of the time is dedicated to providing proofs. These are often technical in nature and therefore there is a continuous back and forth referring to previous steps.

Nonetheless, in my experience it is often useful to use animations to explain new concepts. Something I would like to do along this redesign process is try to find points during the lecture in which animations might clarify what is happening. This would become then part of the teaching script.

Similarly, participation in math classes is often low and in-class activities are usually non-existent. An important part of the redesign involves identifying when such activities might benefit the lecturing process. These will be described in the teaching scripts.

2.2 First version of the redesign (20th May 2019)

As stated in my original goal, I decided to redesign the notes of my sessions, having the lecture notes Federica and Fabian had written as a starting point. After studying the material they had, I set myself three objectives:

- The notes had to be self-contained (originally they were more of a skeleton of the material to be covered).
- The material to be covered had to be lengthened them in block (e.) (which in previous years had been assigned only one session as opposed to three).
- The notes had to be suitable for self-study (by working out examples and adding more details to proofs and the general structure).

2.2.1 Output

I wrote the **notes** as I was teaching the material, producing a document of around a 100 pages (the original notes being around 30 pages). In addition to this, I developed **8 problem sheets**, one for each session I taught.

I have also written a **new version of the Course Guide** (see Appendix I). There are aspects I would like to improve by adding:

- A section in which the workload (in class and at home) is broken down.
- Another (or the same) section explaining the structure of the assignments. This is something I still want to think about, based on the problems I observed this year (see the next section). I think a better assignment scheme is possible (the current structure has weekly assignments that seem to be quite demanding for the students).

2.2.2 Issues encountered

As I taught the course, I discovered that certain things did not work as well as I had expected while preparing the notes. I list them here, arranged depending on their type. Additionally, I outline a possible way of addressing these problems.

Misconceptions about the background of the students

a. At the very beginning there was some confusion regarding the definition of orientation. **Proposed solution:** Time should be allotted to address this in class with an in-depth explanation.

b. Students do not know how de Rham and singular (co)homology relate, even if they know them separately. **Proposed solution:** One of the exercises from Sheet 7 could be used as an in-class activity to explain this.

c. Foliations (and, more generally, tangent distributions) appear at several points during the course. In this iteration, almost half of the attendees were unfamiliar with them. **Proposed solution:** A chapter must be added to the notes to address this in detail; preferably right after block (b.) is completed.

d. There should be a session in which Lie groups are discussed properly (at the moment they appear both in blocks (1) and (4) without providing prior context). **Proposed solution:** Give an assignment for them to review these ideas before the session in which they will be used.

Structuring of the notes

e. Symplectic reduction is presented in block (4) within the context of Hamiltonian Lie group actions, even though it is needed in block (3). **Proposed solution:** State it abstractly in full generality for presymplectic manifolds earlier in the course, as part of block (3).

f. Across several sessions I explained some basic tools (based on volume) for proving that two symplectic manifolds are not symplectomorphic. This should be approached in a more streamlined fashion with a clear goal, as opposed to being isolated examples.

g. Some of the sessions lacked a clear punchline (an obvious learning objective to aim for). The roadmap should be clear to the students and during class more emphasis should be put into the important results. **Proposed solution:** Structure the notes/teaching script following a "red thread", pinpointing the learning objective in each session (and in each hour of the session).

The topics are spread too thinly

The following issues can be simply summarised: too many topics are covered, and it is therefore difficult for the students to achieve proficiency in them. **Proposed solution:** blocks (4) or (5) should be discarded to allow for a deeper look at blocks (1) to (3).

In particular:

h. Many fundamental results regarding linear algebra had to be skipped. Some examples: diagonalisation of symplectic matrices, homogeneous spaces (Lagrangian grassmannian, space of linear symplectic structures), contractibility of the space of compatible almost complex structures, Maslov cycle.

i. We were unable to study submanifolds of symplectic manifolds in depth (despite being a learning objective of block (3). This was aggravated by item (e.).

j. The students are unable to appreciate the parallels between blocks (3) and (5) because they did not remember the key facts from block (3). **Proposed solution:** If we keep block (5), it should be moved to before block (4) or we should give the students an assignment to recall the key parts of block (3) before starting block (5).

The exercises are not conductive to learning

The main problem, outlined in the list below, was that some of the students had not absorbed the material properly before they were presented with non-trivial exercises. **Proposed solution:** Develop exercise sheets that start from the basics, in which their recollection of the definitions and basic lemmas is tested. Then, eventually build up to exercises that complement the material explained in class. A potential idea would be to have them work on a harder problem at home (which is not graded) and then discuss that problem during the lecture.

k. In two occasions, it was unclear to the students why some of the exercises in the sheets were relevant. The reason was that they covered material *complementing* the lectures as opposed to *reinforcing what they had seen.*

l. During the student reviews it was pointed out that the students missed having proper tutorials (i.e. problem solving sessions). Some problem solving was done during the classes themselves, and during the last half hour of each session we had an informal tutorial in which they could ask questions.**Proposed solution:** a more structured format would be better. It would be interesting to have the tutorial during the first hour, where we look at the exercises of the previous week.

m. In-class activities were effective when enough time was dedicated to them. However, they do not work properly when the problems are too difficult.

n. The exercises require the students to invest a lot of time and stress (since they are weekly), but they count only for 15% of the mark. **Proposed solution:** Have instead a few larger assignments to which they can dedicate 2-3 weeks. In-between, have non-graded homework to help them prepare for the next session.

o. The students focus on handing-in the exercises and they ignore the rest of the material. **Proposed solution:** If some of the material cannot be properly evaluated through activities, it should not be part of course. Keep this in mind in the next iteration of the notes.

Other issues

p. One of the goals of my original proposal was to introduce animations in the teaching (since, during talks, I have been able to use them quite effectively). Due to a lack of time I was unable to do it.

q. I should be careful with my boardwriting: Larger hand-writing, use of colours to highlight important results (as in the teaching script).

2.3 First steps towards a new version (6th June 2019)

I have started working on producing a **teaching script**, based on the notes I wrote. Looking at the issues pinpointed above, I decided that the script had to be subject to the following constraints:

- **Timing.** Each section corresponds to an hour of class (based on my observations during this iteration).
- **Red thread I.** Each chapter has a clear goal and a clear path to it.
- **Red thread II.** The relationship between the different chapters is clearly outlined.
- **Activities.** In-class activities are scheduled and structured. They have clear learning objectives, related to the goal of the chapter.
- **At home.** Additional exercises are presented at the end of the chapter. It is marked whether they reinforce the material or they present additional ideas to think about.

2.3.1 Output

In this document I present (in Appendix II), a sample version of the **script for Session 15**. My current goal is to turn the notes in their entirety into a teaching script following the format shown there.

Appendix I

Redesigned course guide for the Mastermath Symplectic Geometry

Why symplectic structures?

A *symplectic structure* is simply a closed non-degenerate 2-form. Much like a Riemannian metric measures distances and angles, a symplectic form measures *areas*. You may think of the closedness condition as an analogue of the notion of flatness for a metric.

It turns out that symplectic structures appear in many settings that may be already familiar to you:

- They have their origin in *classical Mechanics*: The canonical symplectic form in the cotangent bundle of a manifold provides the intrinsic formulation of the Hamiltonian formalism.
- They appear in *affine and projective Algebraic Geometry*: Both $\mathbb{C}^n$ and $\mathbb{CP}^n$ possess canonical symplectic structures which restrict nicely to their subvarieties. Many seemingly algebraic properties of a variety turn out to be symplectic.
- They play an important role in the study of the *Smooth Topology* of 4-manifolds: For instance, they interact in particularly meaningful ways with gauge theory invariants arising from physical theories like Seiberg-Witten.
- They are fundamental in *String Theory*: One of the incarnations of Mirror Symmetry is essentially a duality between Symplectic and Algebraic Geometry.

Due to these profound interactions with other areas, Symplectic Geometry and Topology borrows from a wide spectrum of techniques and tools: Algebraic Geometry, Complex Singularity Theory, Complex Analysis, Differential Topology, Differential Geometry, Dynamical Systems, Partial Differential Equations, Homological Algebra, and others.

Beyond its relevance in other fields of Mathematics, the study of symplectic structures presents very interesting unique behaviours which arise from the so–called *rigid/flexible dichotomy*:

- Powerful theories like pseudoholomorphic curves, generating functions, and microlocal sheaves (which range from being analytical in nature to being purely algebraic) can be used to build invariants.
- Methods from differential topology and topology of PDEs, which we call *h*-principles, can be used to provide constructions.

In the last few years we have seen many new developments regarding both rigidity and flexibility. This makes Symplectic Topology quite a young and active area of research!

What will you learn?

The course we will cover the foundations of Symplectic Geometry, eventually getting into more topological aspects. Along the way, you will discover the answer to the following questions:

- What is a symplectic structure and what is its role in Mechanics?
- How do symplectic manifolds look like locally?
- When do we say that two symplectic structures are "the same"?
- What types of submanifolds does a symplectic manifold have?
- What is an automorphism of a symplectic manifold?
- How do Lie groups act on symplectic manifolds?
- What are contact manifolds and why do they appear as boundaries of symplectic manifolds?
- What types of submanifolds does a contact manifold have?
- How do we classify contact and symplectic structures? How are these two problems related?

- How is Symplectic Geometry different from other geometries?

In particular, after completing the course, you should be able to:

- Identify the different types of subspaces of a symplectic vector space.
- Identify the different types of submanifolds of a symplectic manifold.
- Prove in simple cases that two symplectic manifolds are not symplectomorphic.
- Apply symplectic reduction to a symplectic manifold endowed with a Hamiltonian Lie group action.
- Phrase simple mechanical systems in the language of Symplectic Geometry.
- Use the Moser path method to construct local models and prove stability results.
- Manipulate knots in contact 3-manifolds through their projections.
- Apply h-principle ideas to construct geometric structures in simple settings.

What should you know already?

This course is aimed at the student that has already had a course on Differential Geometry. As such, we expect familiarity with the following notions:

- Manifolds, smooth maps, submanifolds, immersion and submersion theorems.
- Tangent bundle, vector fields, Lie derivatives, flows.
- Cotangent bundle, differential forms, de Rham cohomology.

The following topics will be reviewed during the course, but having some prior knowledge is probably helpful:

- Lie groups and Lie algebras.
- Distributions, involutivity and integrability, foliations and the theorem of Frobenius.
- Classical Mechanics.

A suitable reference for Differential Geometry is:

- J. Lee, Introduction to Smooth Manifolds, second edition Graduate Texts in Mathematics, Springer, 2002.

The relevant chapters are: 1-5,7-12,14-17,19,21.

Which books will we follow?

These notes cover all the material that we will see during the course. Nonetheless, there are three classic references that you may find useful:

- Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology* (third ed.). Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2017.
- Ana Cannas da Silva, *Lectures on symplectic geometry.* Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001.
- Hansjoerg Geiges, *An introduction to contact topology.* Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.

You should also keep an eye on the Mastermath ELO website of the course, as well as Fabian's website (https://www.staff.science.uu.nl/ zilte001/). We will upload there the weekly exercise sheets, as well as any new version of the notes.

Appendix II

Redesigned teaching script (for Session 15)

In this session:

- We will become more familiar with 3-dimensional contact structures. In particular, we will prove a useful criterion to construct them (Proposition 4.5).
- We will look at curves tangent to contact structures, which are called Legendrian knots (Definition 4.7). In particular, we will provide a constructive method that produces many examples (Proposition 4.17).
- We will introduce a tool (Definition 4.18) to help us distinguish Legendrian knots.

3-dimensional contact structures

Until fairly recently (with a few exceptions) Contact and Symplectic Topology had mostly developed in dimensions 3/4. In these dimensions it is possible to have a good geometrical intuition by simply drawing what is happening.

4.1.1 Examples (25 minutes)

Let us provide some explicit examples of globally defined contact structures on 3–manifolds. They all can be shown to be contact by checking the condition $\alpha \wedge d\alpha \neq 0$.

In class activity (25 minutes): For each of the following examples (I will work out the first one myself in the board):

- Check that the given plane field is a contact structure (by looking at the condition $\alpha \wedge d\alpha \neq 0$).
- Draw the coefficients of the second vector field in the framing as a planar curve.

Try to find a pattern: What do all these curves have in common? Hint: compare their position and velocity.

Example 4.1. The structure

$$(\mathbb{R}^3, \xi_{\mathrm{std}} = \ker(dy - z dx))$$

performs almost half a turn with respect to the line field $\langle \partial_z \rangle$. It is (globally) diffeomorphic to

$$(\mathbb{R}^3, \xi_{\mathrm{std}'} = \ker(\cos(z)dx + \sin(z)dy)),$$

which turns infinitely many times with respect to the line field $\langle \partial_z \rangle$. It is also diffeomorphic to

$$(\mathbb{R}^3, \xi_{\mathrm{std}''} = \ker(dz + y dx - x dy)),$$

which performs almost a $\pi/2$-turn with respect to the radial vector field $x\partial_x + y\partial_y$. This structure can be rewritten in cylindrical coordinates (r, θ, z) as

$$\xi_{\mathrm{std}''} = \ker(dz - r^2 d\theta).$$

We simply say that all of them are the **standard contact structure** in $\mathbb{R}^3$. We invite the reader to provide explicit contactomorphisms between all of them. ⧫

Example 4.2. The structure

$$(\mathbb{R}^3, \xi_{\mathrm{OT}} = \ker(\cos(r)dx + \sin(r) r d\theta))$$

turns infinitely many times with respect to the line field $\langle \partial_r \rangle$. It is **not** diffeomorphic to the standard contact structure, and it is called the **contact structure overtwisted at infinity**. See Theorem [-] below and the subsequent discussion. ⧫

Example 4.3. The structures

$$(\mathbb{T}^3, \xi_k = \ker(\cos(\pi k z)dx + \sin(\pi k z)dy)) \qquad k \in \mathbb{Z}^+,$$

turn $k/2$ times with respect to the line field $\langle \partial_z \rangle$. They are not diffeomorphic to one another. The structures are coorientable if and only if k is even. ⧫

Example 4.4. Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. It is defined as the level set $f^{-1}(1)$ of

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

As such, its tangent space is the kernel of the 1-form

$$df = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2).$$

The complex tangencies (i.e. the vectors v such that both v and iv are in $T\mathbb{S}^3$) are simply the complex lines:

$$\begin{aligned}\xi_{\text{std}} = T\mathbb{S}^3 \cap i(T\mathbb{S}^3) = \ker(df) \cap \ker(df \circ i) &= \ker(\lambda_{\text{can}}) \\ &= \ker(-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2) \subset T\mathbb{S}^3.\end{aligned}$$

Which we already saw in the previous session. $(\mathbb{S}^3, \xi_{\text{std}})$ is the compactification of $(\mathbb{R}^3, \xi_{\text{std}})$. We leave this to the reader. ⧫

4.1.2 The contact condition amounts to turning (25 minutes)

The previous examples lead us thus to the following characterisation of the contact condition:

Proposition 4.5. *Fix coordinates (x, y, z) in $S \times [-1, 1]$, where S is a disc, a 2-torus, or a cylinder. Given a plane field of the form*

$$\xi = \ker(\alpha), \qquad \alpha = f dy + g dx,$$

where $f, g : S \times [-1, 1] \to \mathbb{R}$, we may look at the curves:

$$\gamma_{x_0,y_0} : [-1, 1] \to \mathbb{S}^1$$

$$\gamma_{x_0,y_0}(z) = \frac{(f(x_0, y_0, z), g(x_0, y_0, z))}{|f, g|}.$$

Then:

- *ξ is contact at the point (x_0, y_0, z_0) if and only if γ_{x_0,y_0} is an immersion at time z_0.*
- *ξ is involutive if and only if the curves γ_{x_0,y_0} are constant.*

Proof. Indeed, we check that

$$\alpha \wedge d\alpha = [f(\partial_z g) - g(\partial_z f)] dx \wedge dy \wedge dz.$$

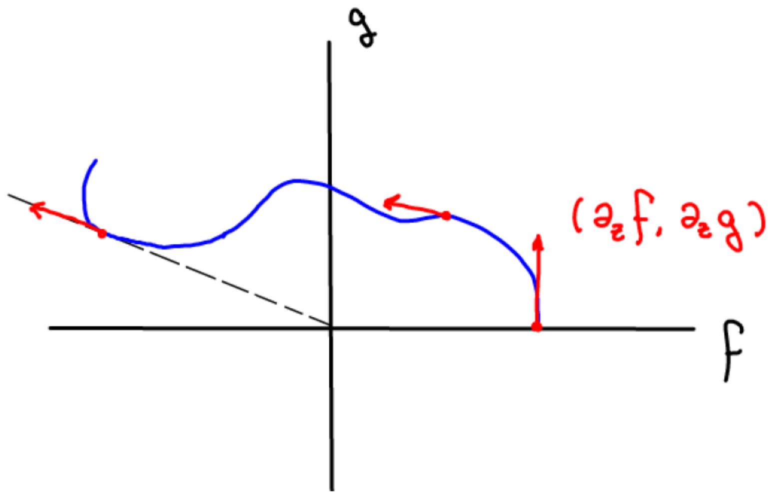

Figure 4.1: A family of curves (f, g) describing a plane field (but we only draw one of them). The points in which (f, g) is colinear with its velocity $(\partial_z f, \partial_z g)$ correspond to singular points of the normalised map $(f, g)/|f, g|$, as seen on the left hand side.

The condition $f(\partial_z g) - g(\partial_z f) \neq 0$ is equivalent to (f, g) and $(\partial_z f, \partial_z g)$ being linearly independent vectors in $\mathbb{R}^2$. This is precisely the immersion condition for $(f, g)/|f, g|$; see Figure 4.1.

□

This is usually phrased as follows: ξ is contact if and only if it turns with respect to any line field tangent to it. This lemma will be extremely useful, because it will allow us to construct contact structures by taking any plane field and "adding to it a bit of turning". Introducing turning can be done by working locally, thanks to the model produced by the following lemma:

Lemma 4.6. *Let (M, ξ) be a 3-manifold endowed with a plane field. Fix $p \in M$. Then, there are local coordinates (x, y, z) around p in which*

$$\xi = \ker(dy + g(x, y, z)dx),$$

where g is a locally defined function.

Proof. Pick a non-vanishing vector field Z tangent to ξ, locally around p. We may then choose a locally defined surface S containing p and transverse to Z. By construction, S is transverse to ξ. Either by hand or by invoking Frobenius' theorem, we find local coordinates (x, y) in S such that the line field $\xi \cap TS$ is spanned by ∂_x. Consider now the flow ϕ_t of Z. We give coordinates (x, y, z) to the point $\phi_z(x, y)$. See Figure 4.2.

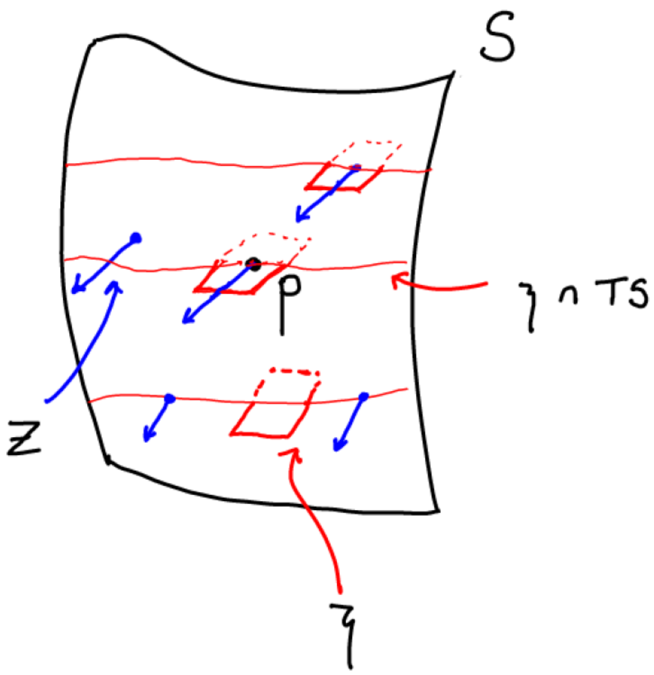

Figure 4.2: Construction of the neighbourhood of p in which ξ is in normal form. Z is a vector field transverse to ξ, S is a surface transverse to it and passing through p.

By construction ξ is tangent to ∂_z in these local coordinates. Additionally, it is tangent to ∂_x at $\{z = 0\}$. These two conditions imply the local form claimed. □

Legendrian knots I

4.2.1 Review of Smooth Knot Theory (15 minutes)

In 3-dimensional Smooth Topology, a knot is an embedding of $\mathbb{S}^1$ into a 3-manifold N. This notion is fundamental due to its role in the definition of surgery (i.e. cutting N along a knot and filling the hole in order to obtain a new manifold). One often focuses on the case in which N is $\mathbb{R}^3$ or $\mathbb{S}^3$ (and this is what we henceforth do).

The simplest knot is the **unknot**. This is the embedding, unique up to isotopy, which is the boundary of an embedded disc. Knot theory consists of determining whether

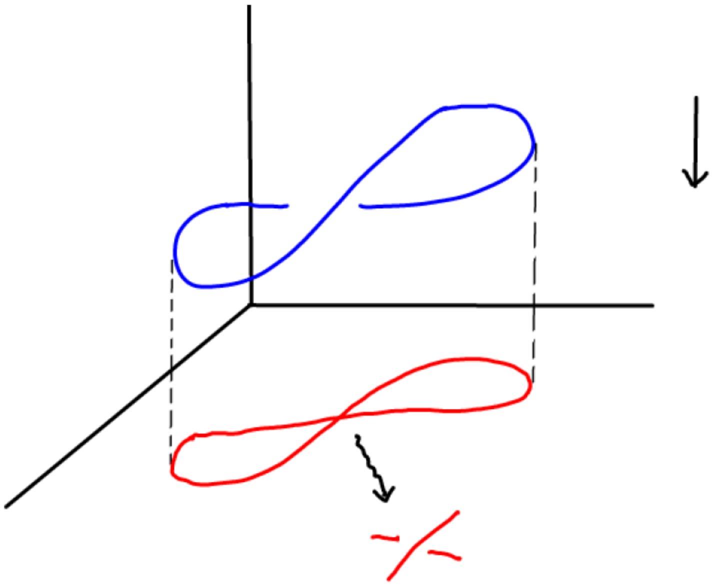

Figure 4.3: An unknot (in blue) and its projection to the horizontal plane (in red). We need to specify which is the over-pass at the point in which the projection has a self-intersection; this is depicted next to the projection with an arrow.

two knots are isotopic to one another. Even the task of determining whether a given knot is in fact the unknot is non-trivial.

The way in which one presents a knot is through a projection. That is, we pick a plane in $\mathbb{R}^3$ and we project the knot to it orthogonally. Such a projection is, generically (i.e. for most choices of plane), an immersed curve with self-intersections. These intersections are an artifact of the projection, and to distinguish them we draw the strands meeting at the intersection as an under-pass and as an over-pass. See Figure 4.3.

We are interested in classifying knots up to isotopy. As we isotope a knot, its projection varies, but (generically) it does so in a controlled way: Only three events, called the Reidemeister moves, may take place. They are depicted in Figure 4.4.

4.2.2 Legendrian Knots (5 minutes)

In a contact 3-manifold we can look at knots as well:

Definition 4.7. *Let (N^3, ξ) be a 3-dimensional contact manifold. A **Legendrian knot** is an embedding $\mathbb{S}^1 \to N$ which is everywhere tangent to ξ.*

A Legendrian knot is a Legendrian in the general sense.

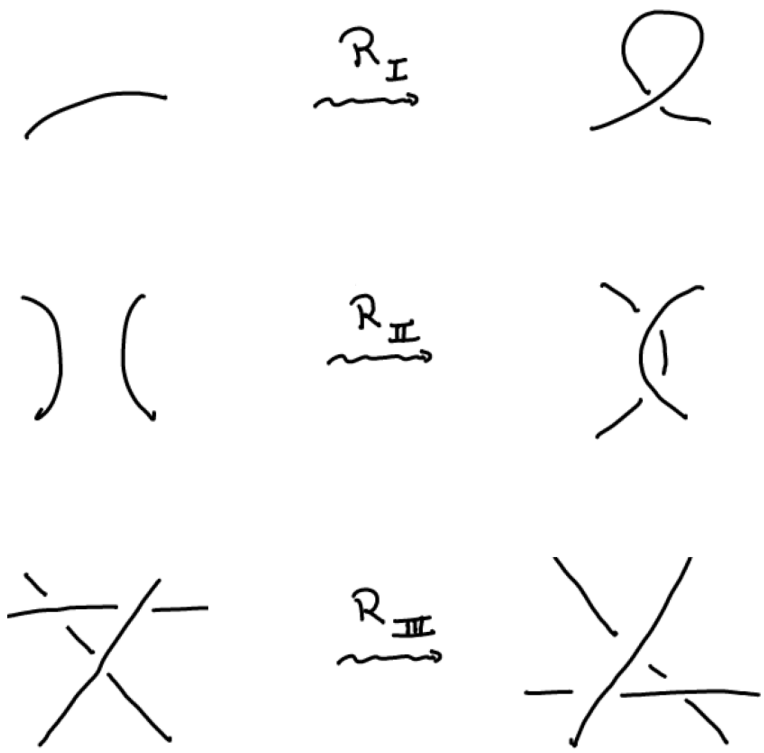

Figure 4.4: The three elementary events one might see as a knot is isotoped. They are called the first, second, and third Reidemeister moves, respectively.

In the contact setting we do not project to an arbitrary 2-plane. Instead, there are two projections that are well-suited to manipulating Legendrians:

Definition 4.8. *Consider* $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$.

- *The map* $\pi_f : \mathbb{R}^3(x, y, z) \to \mathbb{R}^2(x, y)$ *is called the* ***front projection***.
- *The map* $\pi_L : \mathbb{R}^3(x, y, z) \to \mathbb{R}^2(x, z)$ *is called the* ***Lagrangian projection***.

4.2.3 Front projection (30 minutes)

One can completely recover (up to shift in the Lagrangian case) a Legendrian knot from either of its projections. Let us work this out first in the front projection:

Lemma 4.9. *Let* $\gamma : [0, 1] \to \mathbb{R}^3$ *be a Legendrian curve. Suppose that* $\pi_f \circ \gamma(t) = (x(t), y(t))$ *is immersed. Then the missing* z*–coordinate can be recovered using the expression:*

$$z(t) = \frac{dy}{dx}(t).$$

Proof. Since $\pi_f \circ \gamma(t)$ is immersed and $dy - zdx$ evaluates to zero on γ, we deduce that $\gamma^* dx = x'(t)dt$ is nonzero. Then we can solve $z(t) = dy/dx$, as claimed. □

Remark 4.10. A particular case is a curve of the form $(t, y(t), z(t))$. It must satisfy $y'(t) = z(t)$. ♦

Now, not all Legendrian curves γ in $(\mathbb{R}^3, \xi_{\text{std}})$ project to an immersed curve $\pi_f \circ \gamma$:

Lemma 4.11. *The curve* $\gamma : \mathbb{R} \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (x(t) = t^2/2, y(t) = t^3/3, z(t) = t)$$

is embedded and Legendrian.

Proof. The front projection $\pi_f \circ \gamma(t) = (x(t) = t^2/2, y(t) = t^3/3)$ has a singularity (i.e. fails to be immersed) at $t = 0$, which we call the **cusp**. This is the simplest singularity a planar curve may have.

The curve γ itself is embedded, since the map $z(t)$ is a diffeomorphism of $\mathbb{R}$. For the Legendrian condition, it is sufficient to show that $\gamma^*(dy - zdx) = t^2 dt - t^2 dt = 0$. □

One can prove that:

Proposition 4.12. *Let γ be a Legendrian knot. After a (C^∞) small perturbation, it may be assumed that $\pi_f \circ \gamma$:*

- *Fails to be an immersion at a finite collection of points.*
- *At these points it is equivalent to the cusp or its mirror image.*

Proof. This statement requires transversality theory, which is beyond the scope of this course. The idea is roughly the following: $\pi_f \circ \gamma$ fails to be an immersion if and only if γ is tangent to the projection direction $\langle \partial_z \rangle$. When this happens, since γ itself is immersed, we have that γ must be graphical over its z–coordinate. That is, up to reparametrisation we may take $\gamma(t) = (x(t), y(t), t)$.

Transversality tells us that one can perturb γ so that these tangencies are as simple as possible. In this case, this means that they should be cuadratic so $x(t)$ should agree with $\pm t^2$ (up to reparametrisation in the domain and the target). The y coordinate is uniquely determined from x and z (by integrating), yielding $y(t) = \frac{2t^3}{3}$, i.e. the cusp. $\square$

Apart from cusps, the planar curve $\pi_f \circ \gamma$ may fail to be embedded:

Lemma 4.13. *Let γ be a Legendrian knot. Two branches of $\pi_f \circ \gamma$ meet at an intersection point with different slopes.*

Proof. Suppose there are two distinct times t_0 and t_1 such that

$$\pi_f \circ \gamma(t_0) = (x(t_0), y(t_0)) = (x(t_1), y(t_1)) = \pi_f \circ \gamma(t_1).$$

Embeddedness of γ implies that $z(t_0) \neq z(t_1)$. This can be rewritten as:

$$\frac{dy}{dx}(t_0) = z(t_0) \neq z(t_1) = \frac{dy}{dx}(t_1)$$

i.e. the regions of $\pi_f \circ \gamma$ for times close to t_0 and for times close to t_1 have different slope, as claimed. $\square$

In particular: At a crossing we do not need to specify whether it is an underpass or an overpass, because this is given by the slope. See Figure 4.5.

One can show (appealing again to transversality) that:

Proposition 4.14. *Let γ be a Legendrian knot. After a (C^∞) small perturbation, it may be assumed that $\pi_f \circ \gamma$:*

- *Has only finitely many self-intersections.*
- *At each intersection point only two branches meet.*

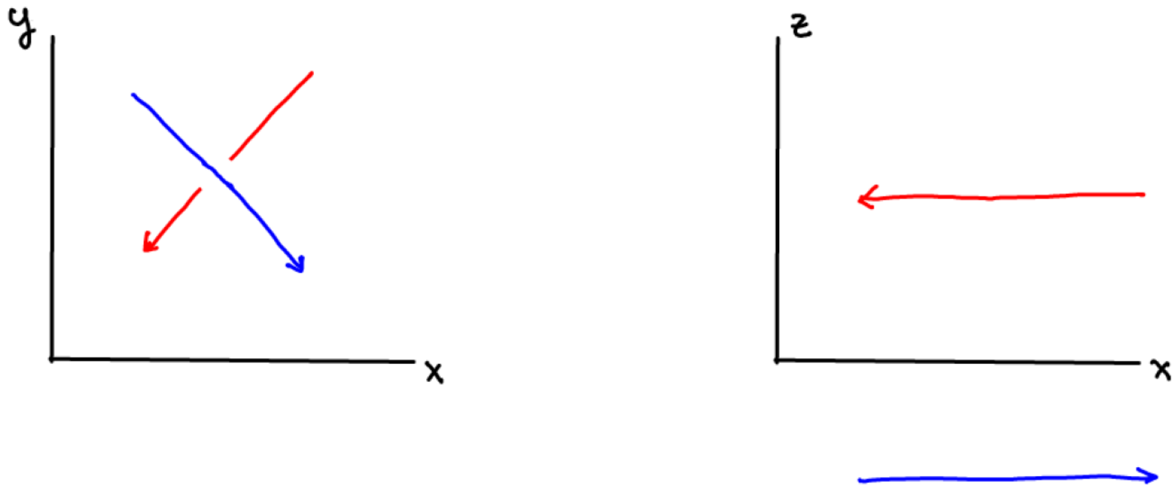

Figure 4.5: On the left we depict two curves whose front projections intersect. The slopes of the strands are constant and different, so their z–coordinates are different. In particular, the self-intersection of the front does not correspond to a self-intersection in 3-dimensional space.

In class activity (10 minutes): Given the left-hand side of Figure 4.6, draw the right hand-side. Hints:

- Look first at the cusps, what do they correspond to on the (x, z)–coordinates?
- Look now at the maxima of the upper strand and the minima of the lower one, what should they correspond to?
- Follow the slope of the upper and lower branches. When are they positive? When are they negative?

Legendrian knots II

4.3.1 Lagrangian projection (20 minutes)

Let us work in the Lagrangian projection now:

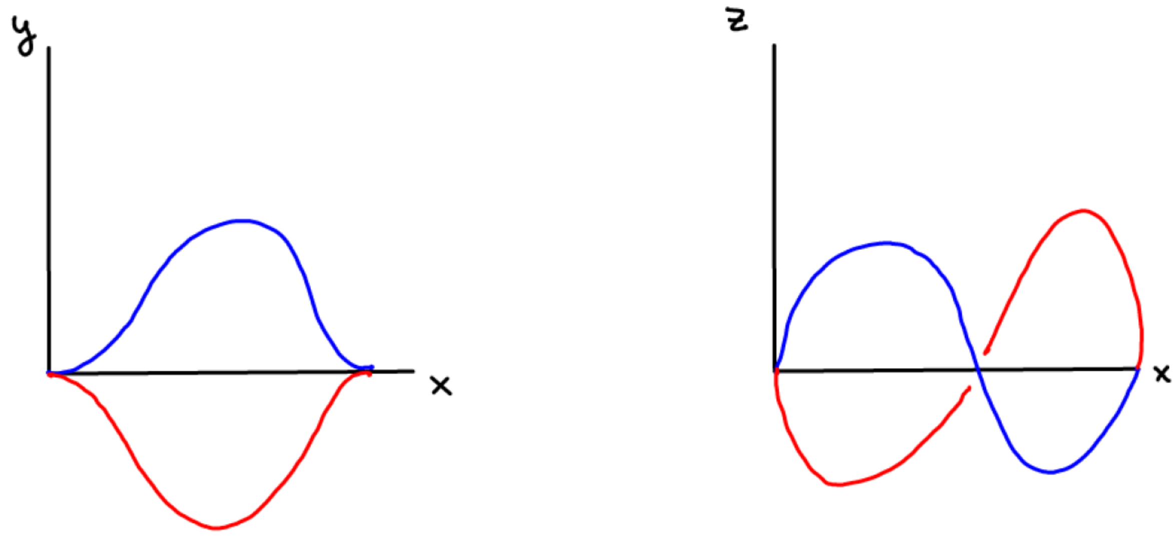

Figure 4.6: The so-called Legendrian unknot (which is in particular a smooth unknot). We draw it in both projections. In the front (left) it has two cusps. In the Lagrangian projection (right) we see a self-intersection.

Proposition 4.15. *Let $\gamma : [0,1] \to \mathbb{R}^3$ be an immersed Legendrian curve. Then:*

- *Its Lagrangian projection $\pi_L \circ \gamma(t) = (x(t), z(t))$ is an immersed planar curve.*
- *The missing coordinate can be recovered by integrating:*

$$y'(t) = z(t)x'(t), \qquad y(t) = y(0) + \int_0^t z(s)x'(s)ds.$$

Proof. The curve $\pi_L \circ \gamma$ would fail to be immersed at time t if at only if γ is tangent to ∂_y, the direction of projection. Since γ is Legendrian and immersed, this can never happen (because ∂_y is not tangent to ξ_{std}).

The second claim follows because $\gamma^*(dy - zdx) = 0$, due to the Legendrian condition. □

In particular:

Corollary 4.16. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ be an embedded Legendrian curve. Then:*

- *Its Lagrangian projection $\pi_L \circ \gamma$ bounds zero area.*
- *Let $t_0, t_1 \in \mathbb{S}^1$ be times at which $\pi_L \circ \gamma(t_0) = \pi_L \circ \gamma(t_1)$. Then the curve $\pi_L \circ \gamma|_{[t_0,t_1]}$ does not bound zero area.*

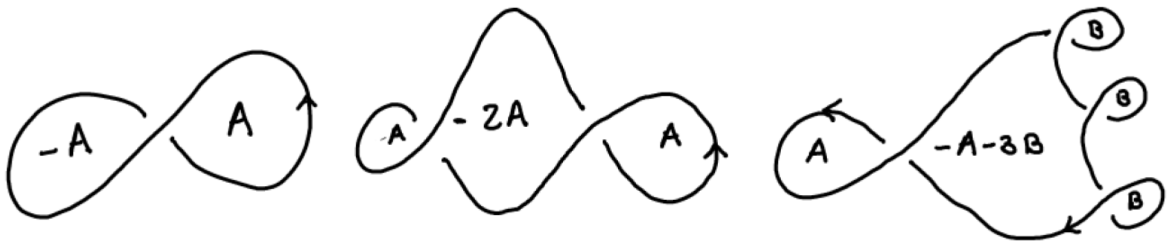

Figure 4.7: The Lagrangian projection of 3 Legendrians. In every region we specify its area, the total sum of which must add to zero. The area of each region additionally allows us to compute which are the over-passes.

Proof. See Figure 4.7.

For the first statement: Since γ is a closed curve, it bounds a (possibly not embedded) disc D. Then we may apply Stokes to show that:

$$0 = \int_\gamma dy = \int_\gamma z dx = \int_{\pi_L \circ \gamma} z dx = \int_{\pi_L(D)} dz dx,$$

where the first equality follows from the fact that γ is closed.

For the second statement we have that $\pi_L \circ \gamma|_{[t_0,t_1]}$ is a closed planar curve, so it bounds a (possibly non-immersed) disc D. Then we have:

$$0 \neq y(t_1) - y(t_0) = \int_{\gamma|_{[t_0,t_1]}} dy = \int_{\gamma|_{[t_0,t_1]}} z dx = \int_D dz dx,$$

where the first inequality is due to the embeddedness condition. □

4.3.2 Construction of Legendrians (20 minutes)

The first meaningful question one might pose is: how rich is Legendrian Knot Theory? For instance, can any smooth knot be represented by a Legendrian knot? The answer is yes:

Proposition 4.17. *Let γ be a smooth knot in $\mathbb{R}^3$. Then, it is smoothly isotopic to a Legendrian knot $\widetilde{\gamma}$ (which is not unique!).*

Proof. This is a proof by picture. It can be done in either projection. See Figures 4.8 and 4.9.

□

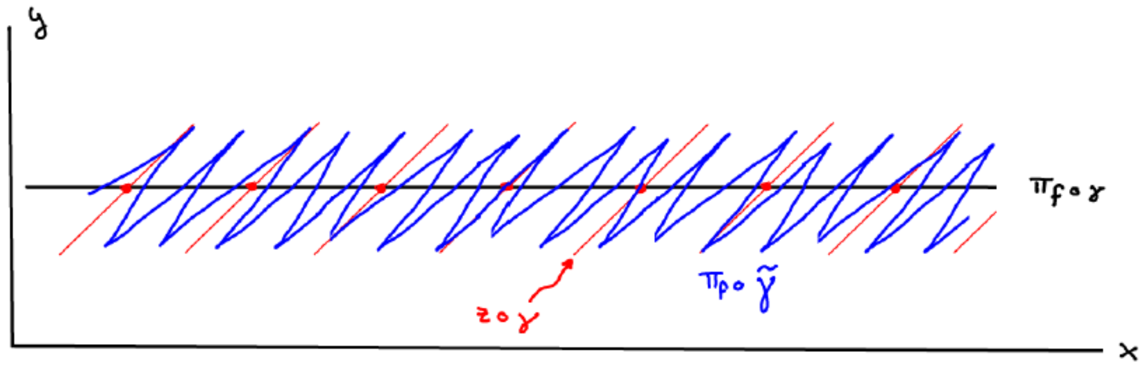

Figure 4.8: We describe a non-Legendrian curve γ in 3-space by looking at its front projection (in black) and specifying its missing z–coordinate by drawing a slope (in red). We claim that we can approximate it by a Legendrian curve. Indeed, we draw a curve (in blue) whose front projection is close to the black curve and whose slope is very close to the red slopes. At the turning points it has cusps. Its unique lift is the desired Legendrian $\tilde{\gamma}$ approximating γ.

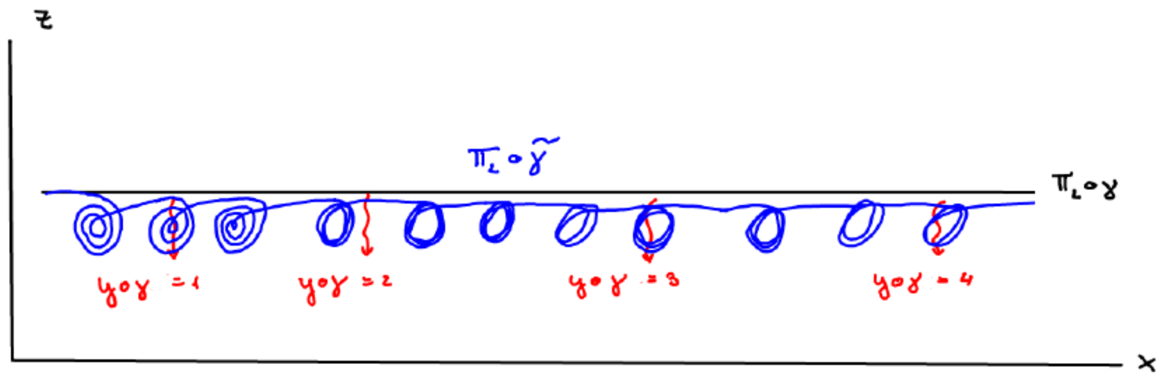

Figure 4.9: We describe a non-Legendrian curve in 3-space by looking at its Lagrangian projection (in black); we must keep track of the missing z–coordinate (which is a number at each point of the curve, recorded in red). We draw a blue curve which is very close to the black curve and that has many loops. The area of these loops accounts exactly for the desired displacement in z. In this manner, its unique lift is the Legendrian that we desired. The self-intersections that appear when we introduce loops do not lift to actual self-intersection (because the loops bound positive area), so the Legendrian constructed is embedded.

4.3.3 Rotation number (10 minutes)

Legendrian Knot Theory studies the question: when are two given Legendrian knots homotopic to one another (as Legendrian knots)? It is a necessary condition that they are smoothly isotopic, but this is not sufficient. The first additional invariant we can define is:

Definition 4.18. *Let $\gamma : \mathbb{S}^1 \to (\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$ be an immersed Legendrian. Its Lagrangian projection $\pi_L \circ \gamma$ is a closed and immersed planar curve in $\mathbb{R}^2(x,z)$. As such, we may look at the Gauss map:*

$$\rho(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2(x,z)$$

$$\rho(\gamma)(t) = \frac{(\pi_L \circ \gamma)'(t)}{|(\pi_L \circ \gamma)'(t)|}.$$

Then, the ***rotation number*** *of γ is the degree of $\rho(\gamma)$, which is an integer.*

Remark 4.19. Looking at this definition, you should convince yourself that it depends on the orientation we put in $\mathbb{R}^2(x,z)$,. Here we are assuming that it is the standard one given by the basis $\{\partial_x, \partial_z\}$.

Lemma 4.20. *The rotation number is invariant under homotopies of immersed Legendrians.*

Proof. Let $(\gamma_s)_{s\in[0,1]}$ be a family of immersed Legendrians. The corresponding projections $(\pi_L \circ \gamma_s)_{s\in[0,1]}$ are also immersed. As such, we obtain a homotopy of maps:

$$\rho(\gamma_s)(t) = \frac{(\pi_L \circ \gamma_s)'(t)}{|(\pi_L \circ \gamma_s)'(t)|}$$

We conclude by recalling that the degree is a homotopy invariant of maps $\mathbb{S}^1 \to \mathbb{S}^1$. □

We will use this Lemma in the exercises to distinguish Legendrian knots that are smoothly isotopic.

Remark 4.21. To a Legendrian knot one can assign another invariant called the *Thurston-Bennequin number*, which measures the twisting of ξ_{std} with respect to the knot. We will not look into this any further; I invite you to read the book by Geiges. ⧫

Preparation for Session 16

Make sure you are comfortable with the definitions and statements marked as important (in blue) in the notes. Then, take a look at the first exercise of each block below (we will spend the first hour of Session 16 discussing them).

4.4.1 Contact forms, Reeb fields

Exercise 4.1. Prove that the following 1-forms are contact forms in $\mathbb{R}^3$ (in either standard coordinates (x, y, z) or polar coordinates (r, θ, z)). Compute their Reeb vector fields. Describe their closed Reeb orbits (i.e. the orbits of the Reeb vector field which are periodic), computing their periods.

- $\alpha_1 = dy - zdx$,
- $\alpha_2 = \cos(z)dx + \sin(z)dy$,
- $\alpha_3 = dz - ydx + xdy$,
- $\alpha_4 = \cos(r)dx + \sin(r)rd\theta$.

Exercise 4.2. Prove that the following plane fields are contact structures:

$$(\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3, \xi_k = \ker(\cos(\pi kz)dx + \sin(\pi kz)dy)) \qquad k \in \mathbb{Z}^+.$$

Compute the Reeb vector field of the given contact forms. Describe their closed Reeb orbits (with their periods).

Proof. First observe that, for k odd, the forms given are in fact not well-defined at $z = 0, 1$. This tells us that the corresponding plane fields are not coorientable (this showed up in the last exercise of the previous sheet already). This is not a problem when we check the contact condition, which is just a local computation. We write:

$$\alpha_k = \cos(\pi kz)dx + \sin(\pi kz)dy, \qquad d\alpha_k = \pi k(-\sin(\pi kz)dzdx + \cos(\pi kz)dzdy)$$

$$\alpha_k \wedge d\alpha_k = -\pi k dx \wedge dy \wedge dz$$

which is a volume form, so ξ_k is contact.

We now compute the Reeb field. **Important remark**: for k odd, the Reeb field is not well-defined! If a contact structure is not coorientable, it does not make sense to talk about its Reeb field, because the Reeb field is defined in terms of a contact *form*. Now, for k even, the kernel of $d\alpha_k$ is spanned by

$$R_k = \cos(\pi kz)\partial_x + \sin(\pi kz)\partial_y$$

which satisfies $\alpha_k(R_k) = 1$, so it is the Reeb vector field.

A torus $\{z = z_0\}$ is foliated by closed orbits of R_k if and only if $\cos(\pi k z_0)$ and $\sin(\pi k z_0)$ are linearly dependent over the rationals, i.e. either $\cos(\pi k z_0) = 0$ or $\tan(\pi k z_0)$ is a rational number. Otherwise the torus is foliated by orbits which are copies of $\mathbb{R}$. If $\cos(\pi k z_0) = 0$ or $\sin(\pi k z_0) = 0$ the period of the orbits is 1. Otherwise, if $\tan(\pi k z_0) = p/q$, with p, q coprime integers, the corresponding orbits close up for the first time when you move q in the direction of x and p in the direction of y. Since the speed in x is $\cos(\pi k z_0)$, this tells us that the period is $q/\cos(\pi k z_0) = p/\sin(\pi k z_0)$. □

4.4.2 Classification of contact structures

> *Exercise* 4.3. Show that any two plane fields in $\mathbb{R}^3$ are homotopic to one another. Show that the space of plane fields in $\mathbb{S}^3$ has $\mathbb{Z}$ components.

Proof. The cotangent bundle of $\mathbb{R}^3$ is trivial $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. In particular, its projectivisation is trivial too $\mathbb{P}T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{RP}^2$. Giving a plane field in $\mathbb{R}^3$ amounts to giving a section $s : \mathbb{R}^3 \to \mathbb{P}T^*\mathbb{R}^3$ (indeed, such an s has a well-defined kernel at each point, which is the corresponding plane field). Thus, plane fields in $\mathbb{R}^3$ up to homotopy are the same as sections of $\mathbb{P}T^*\mathbb{R}^3$ up to homotopy, i.e. the same as maps $\mathbb{R}^3 \to \mathbb{RP}^2$ up to homotopy. Since $\mathbb{R}^3$ is contractible, all of them are homotopic to one another. This also shows that the space of plane fields in $\mathbb{R}^3$ is contractible.

Any closed 3-manifold is parallelisable (this is a non-trivial theorem!) As such, $\mathbb{P}T^*\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{RP}^2$. Plane fields in the 3-sphere are thus described by maps $\mathbb{S}^3 \to \mathbb{RP}^2$. The possible homotopy classes are then given by $\pi_3(\mathbb{RP}^2) = \pi_3(\mathbb{S}^2) = \mathbb{Z}$, where we use that the universal cover of $\mathbb{RP}^2$ is $\mathbb{S}^2$. □

> *Exercise* 4.4. Consider $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$. Show any arbitrarily big (but compact) domain of $\mathbb{R}^3$ can be mapped to an arbitrarily small one by a contactomorphism of ξ_{std}.

Proof. Consider the family of maps $f_\lambda(x, y, z) = (\lambda x, \lambda^2 y, \lambda z)$. Since

$$f_\lambda^*(dy - zdx) = \lambda^2(dy - zdx)$$

we conclude that they are contactomorphisms for every $\lambda \neq 0$. By taking λ sufficiently small, we may map any arbitrarily big compact set in $\mathbb{R}^3$ to a small one. □

The previous exercise can be used to prove the following statement:

Proposition 4.22. $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$ *is contactomorphic to an arbitrarily small ball (also endowed with* ξ_{std}*).*

Proof. The rough idea is that one can construct a contactomorphism $\mathbb{D}^3 \to \mathbb{R}^3$ as the limit of a family of embeddings $\mathbb{D}^3 \to \mathbb{R}^3$ that preserve the contact structure and whose images are progressively bigger. Formalising this statement requires the use of contact Hamiltonians, which we did not cover in the course. □

Exercise 4.5. Let ξ_0 and ξ_1 be contact structures in $\mathbb{R}^3$. Show that they are homotopic (as contact structures) to one another if and only if they induce the same orientation. Hint: use Darboux and think about the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$.

Proof. Suppose that ξ_0 induces the standard orientation. It is sufficient to show that it is homotopic to $\xi_{\text{std}} = \ker(dy + zdx)$, which also induces the standard orientation. Use Darboux' theorem to find a open ball U_0 with coordinates (x', y', z') in which $\xi_0 = \ker(dy' + z'dx')$. We can assume that U_0 is the image on an orientation preserving embedding $\psi_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which is additionally a contactomorphism (by using the Proposition preceding this exercise).

We claim that the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$ preserving the orientation is connected. The intuitive idea is that one can precompose any embedding f with a homotopy of embeddings $(\rho_r)_{r\in(0,\infty]} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\rho_r(\mathbb{R}^3) = \mathbb{D}^3_r$. As ε goes to zero, the map $f \circ \rho_\varepsilon$ sees progressively less and less of f and remembers only the differential of f at the origin. This effectively provides a retraction of the space of embeddings onto $\mathrm{GL}(\mathbb{R}^3)$, which has two components, corresponding to the two orientations.

Assuming this, find a path ψ_t between ψ_0 and the identity $\psi_1 = \mathrm{Id}_{\mathbb{R}^3}$. Thus, the family $(\psi_t)^*\xi_0$ is a homotopy between $(\psi_0)^*\xi_0 = \xi_{\text{std}}$ and $(\psi_1)^*\xi_0 = \xi_0$. □

4.4.3 Legendrians and their front projection

Exercise 4.6. Check that any legendrian $[0,1] \to (\mathbb{R}^3, \ker(dy - zdx))$ which is graphical over the x–coordinate can be reparametrised to be of the form $(x, y(x), y'(x))$, with y a function of x.

Exercise 4.7. As shown in the previous exercise, $(\mathbb{R}^3, \alpha_{\text{can}} = dy - zdx)$ is the space of 1-jets of functions from $\mathbb{R}$ to $\mathbb{R}$. I.e, we think of y as a function of x and z as its derivative.

Consider the maps:

- $f_1(t) = (x(t) = t^2, y(t) = t^3)$.
- $f_2(t) = (x(t) = t^l, y(t) = t^k)$, with $k > l$ positive integers.

Lift them to Legendrians in $\mathbb{R}^3$ (i.e. find expressions $z(t)$ such that $(x(t), y(t), z(t))$ is a parametrised curve tangent to $\ker(\alpha_{\text{can}})$). Which of the resulting Legendrians are immersed?

Proof. We need the expression $y'(t) - z(t)x'(t)$ to hold. For f_2 this means that:

$$kt^{k-1} - lz(t)t^{l-1} = 0, \qquad z(t) = \frac{k}{l}t^{k-l}.$$

As soon as $k \geq l$, this is a well-defined expression. Now, the tangent vector to f_2 is:

$$f_2'(t) = (x'(t), y'(t), z'(t)) = (lt^{l-1}, kt^{k-1}, \frac{k(k-l)}{l}t^{k-l-1})$$

which vanishes at $t = 0$ if and only if $l > 1$ and $k > l + 1$. Otherwise f_2 is immersed (for instance, if $k = l + 1$, as is the case for f_1). □

Exercise 4.8. This is a follow-up of the previous exercise. Lift the following maps to Legendrians in $\mathbb{R}^3$:

- $f_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t s(s^2 - \varepsilon)ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.
- $g_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t (s^2 - \varepsilon)^2 ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.

For which values of the parameter are the resulting Legendrians embedded? Draw their front and Lagrangian projections schematically as ε varies.

The first family is called the **first Reidemeister move**. The second one is called the **stabilisation**.

Proof. We compute as before. For f_ε the expression $t(t^2 - \varepsilon) + z(t)(t^2 - \varepsilon)$ implies that $z(t) = t$. In particular, the curves are immersed for all times. Since $z(t)$ is strictly increasing, it follows that they are embedded too. This implies that the family constructed is a homotopy of embedded legendrians.

For g_ε we solve $(t^2 - \varepsilon)^2 + z(t)(t^2 - \varepsilon)$, yielding $z(t) = t^2 - \varepsilon$. In particular, the curve $g_0 = (t^3/3, t^5/5, t^2)$ has a singular point at $t = 0$. All other curves are immersed because the only critical point of $z(t)$ takes place at $t = 0$, which is not critical for $x(t)$, whose critical points are at $t = \pm\sqrt{\varepsilon}$. Additionally, they are embedded: this follows because $y(t)$ is strictly increasing outside of the origin but $z(t)$ is decreasing for $t < 0$ and increasing for $t > 0$. Thus, the family g_ε is not a homotopy of immersed/embedded legendrians, because a singularity appears at $\varepsilon = 0$.

Use Wolfram Alpha (or something else) to plot these! In each of the projections, determine which is the over-crossing. □

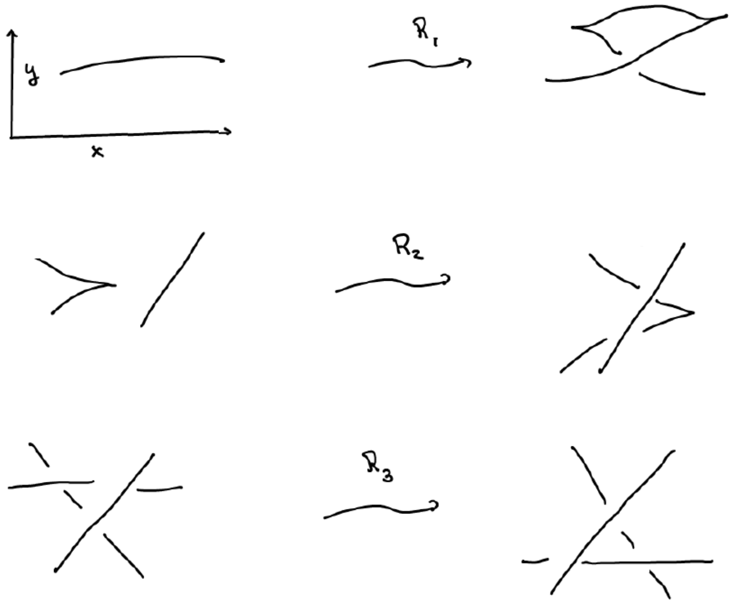

Figure 4.10: The three Legendrian Reidemeister moves in the front projection. The over-crossings represent strands with greater slope and therefore greater z-value.

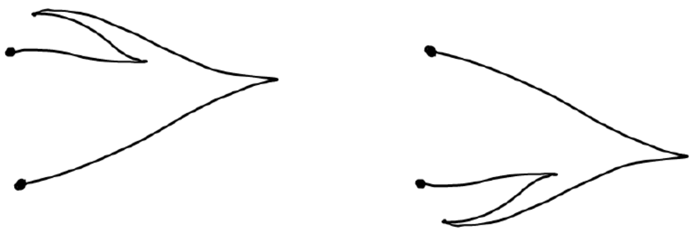

Figure 4.11: Two pieces of Legendrian knot, shown in the front projection.

Exercise 4.9. Using the three Reidemeister moves (Figure 4.10) show that there is a homotopy of Legendrian embeddings connecting the following two local configurations shown in Figure 4.11.

Proof. See Figure 4.12. □

4.4.4 The rotation number

Exercise 4.10. Let $\gamma : \mathbb{S}^1 \to (\mathbb{R}^3, \ker(dy - zdx))$ be a Legendrian knot. Show that the rotation number of $\gamma(-t)$ is minus the rotation number of $\gamma(t)$.

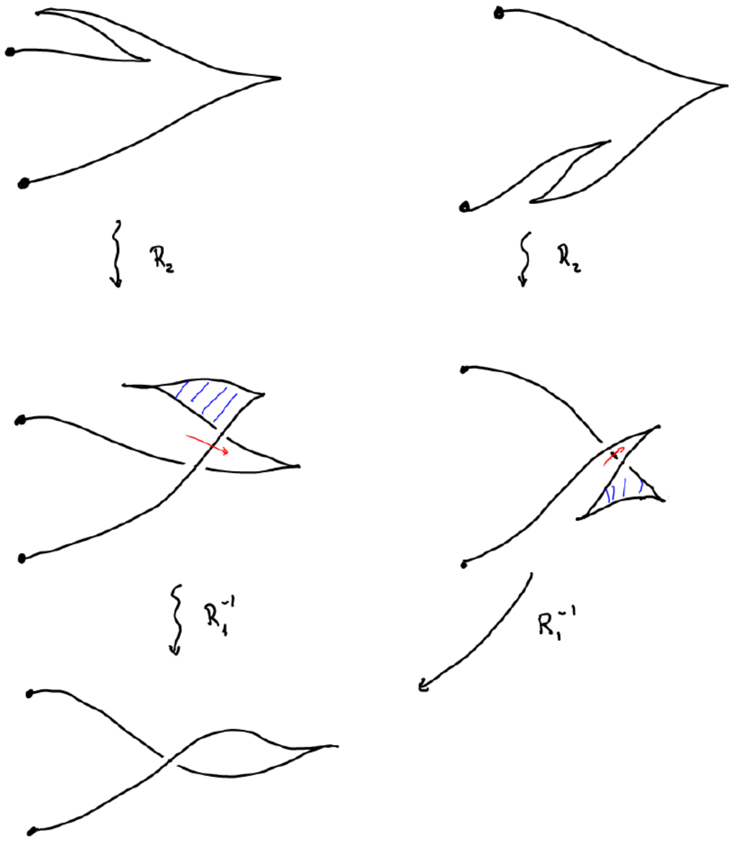

Figure 4.12: The embedded Legendrian homotopy between the two configurations, expressed in terms of Reidemeister moves. The blue areas correspond to Reidemeister I moves.

Exercise 4.11. For each integer $k \in \mathbb{Z}$, find a Legendrian knot in $(\mathbb{R}^3, \ker(dy - zdx))$ with k as its rotation number. To describe the knots, draw schematically their front and Lagrangian projections, stating what convention you use to draw the crossings. It is sufficient that you provide the picture for $k = 0, 1, 2$ and you briefly explain how the general case goes. Remark: you should explain why the two projections you draw indeed correspond to the same knot and you should explain how the rotation is computed from them.

Proof. See Figure 4.13. The main idea is to simply draw curves γ_k in the (x, z)-plane (the Lagrangian projection) bounding zero area and such that γ_k' has degree k. Any such curve will lift to a closed Legendrian, thanks to the formula

$$y(t) = y(0) + \int_0^t zdx,$$

to fix this lift we pick $y(0)$ arbitrarily. According to the previous exercise, it is enough to construct γ_k for $k \geq 0$.

Now, on the left hand side is the unknot, as seen in class. Its Lagrangian projection is a figure eight, which bounds zero area and has rotation zero. The idea now is to add to this Lagrangian projection additional loops: Adding a turn either clockwise or counterclockwise substracts or adds 1 to the rotation number, respectively. This is depicted in pictures two and three. One can make the curve γ_k describe one big lobe in clockwise direction and $k+1$ lobes in counterclockwise direction (the cases depicted are $k = 1, 2$). It is important to make sure that the $k + 1$ lobes bound together the same (unsigned) area as the big lobe, in order to yield a closed curve (this is badly depicted in the picture!).

In order to produce the front projection, we look at the points in which the Lagrangian projection is tangent to the z-direction. These points (marked in the figure) correspond to the cusps of the front. Each strand in-between these points is graphical over the x direction, so we can draw it by recalling that z recovers the slope in the front. In particular: each time we transverse one of the right-most z-tangencies, we are increasing in z, so the corresponding cusp in the front is transversed downwards (because the slope is increasing). Similarly, every time we cross one of the tangencies in-between the small lobes, we are decreasing in slope; thus, the corresponding cusp is also transversed downwards. This tells us that we keep making zig-zags in the front projection.

□

Exercise 4.12. Given $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ Legendrian immersion and a trivialisation of $\xi_{\text{std}} = \ker(dy - zdx)$, there is a map:

$$\rho(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1$$

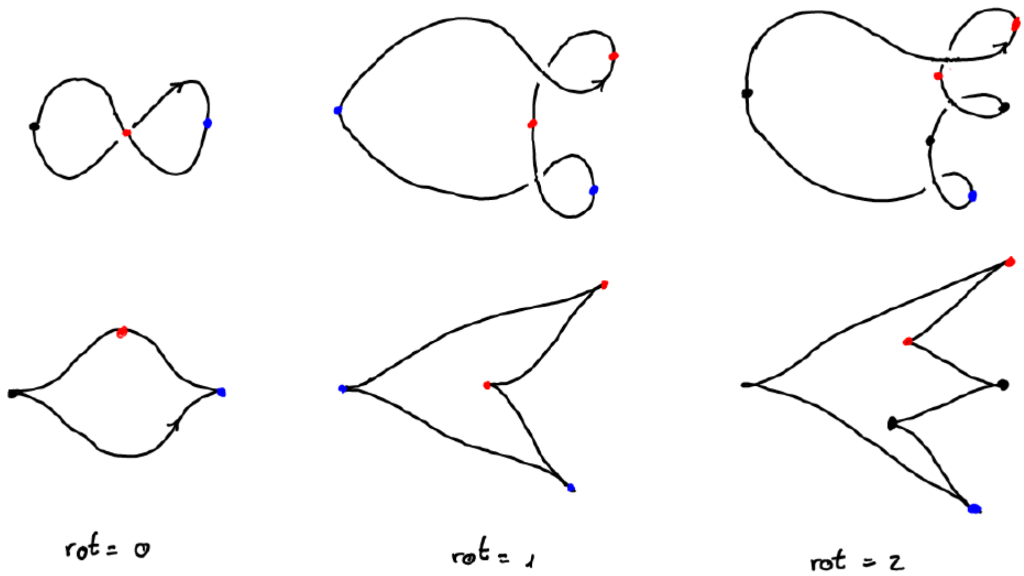

Figure 4.13: Three unknots. The Lagrangian projection on top and the front projection at the bottom. Some of the points are coloured to identify them between the two projections. All the pictures are supposed to be symmetric with respect to the x-axis (the horizontal one in both cases). The over-crossings correspond to greater z and thus to greater slope.

$$\rho(\gamma)(t) = \frac{\gamma'(t)}{|\gamma'(t)|} \in \xi_{\gamma(t)} \cong \mathbb{R}^2$$

where the identification $\xi_{\gamma(t)} \cong \mathbb{R}^2$ depends on the choice of trivialisation. Show that the absolute value of the degree of $\rho(\gamma)$ is independent of the trivialisation. How is this related to the rotation number of γ?

Proof. Trivialisations of the plane field ξ_{std} correspond to framings $\{X, Y\}$ of ξ_{std} (indeed, we map X to ∂_x in $\mathbb{R}^2$ and Y to ∂_y). Such a trivialisation $\xi_{\mathrm{std}} \cong \mathbb{R}^3 \times \mathbb{R}^2$ in particular provides an orientation of ξ_{std} by taking the standard orientation in $\mathbb{R}^2$ on each fibre.

Now, since $\mathbb{R}^3$ is contractible, the space of sections of ξ_{std} is contractible. As such, any two choices of framing $\{X_0, Y_0\}$ and $\{X_1, Y_1\}$ inducing the same orientation are homotopic to one another by a family $\{X_s, Y_s\}_{s\in[0,1]}$. For each s, the corresponding map $\rho_s(\gamma)$ is a planar curve (defined as in the statement, where we indicate the dependence with respect to the trivialisation using the subscript s). The degree of a planar curve is constant in its homotopy class, i.e. since $s \to \deg(\rho_s(\gamma))$ is continuous and takes values in the integers, it is constant. When we consider $\{Y, X\}$ instead of $\{X, Y\}$, the degree changes signs.

The rotation number is defined as the degree computed in the standard trivialisation $X = \partial_x + z\partial_y$ and $Y = \partial_z$.

What you should take from this exercise is that the rotation number can be defined on any contact 3-manifold (M, ξ), in an analogous manner, once we fix a trivialisation of ξ (which is not always possible, because ξ might not be trivial as a bundle). □